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On the global offensive alliance number of a graph[☆]

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Abstract

An offensive alliance in a graph $\Gamma = (V, E)$ is a set of vertices $S \subset V$ where for each vertex v in its boundary the majority of vertices in v's closed neighborhood are in S. In the case of strong offensive alliance, strict majority is required. An alliance S is called global if it affects every vertex in $V \setminus S$, that is, S is a dominating set of Γ. The *global offensive alliance number* $\gamma_o(\Gamma)$ is the minimum cardinality of a global offensive alliance in Γ. An offensive alliance is connected if its induced subgraph is connected. The *global-connected offensive alliance number*, $\gamma_{co}(\Gamma)$, is the minimum cardinality of a global-connected offensive alliance in Γ.

In this paper we obtain several tight bounds on $\gamma_o(\Gamma)$ and $\gamma_{co}(\Gamma)$ in terms of several parameters of Γ. The case of strong alliances is studied by analogy.

Keywords: Offensive alliances; Global alliances; Domination; Independence number

1. Introduction

The study of defensive alliances in graphs, together with a variety of other kinds of alliances, was introduced in [6]. In the cited paper there was initiated the study of the mathematical properties of alliances. In particular, several bounds on the defensive alliance number were given. The particular case of global (strong) defensive alliance was investigated in [4].

The study of offensive alliances was initiated by Favaron et al. in [2] where there were derived several bounds on the offensive alliance number and the strong offensive alliance number. On the other hand, in [7] there were obtained several tight bounds on different types of alliance numbers of a graph: (global) defensive alliance number, global offensive alliance number and global dual alliance number. In particular, there was investigated the relationship between the alliance numbers of a graph and its algebraic connectivity, its spectral radius, and its Laplacian spectral radius. A particular study of the alliance numbers, for the case of planar graphs, can be found in [9]. Moreover, for the study of defensive alliances in the line graph of a simple graph we cite [11].

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The aim of this paper is to study mathematical properties of the global offensive alliance number and the global strong offensive alliance number of a graph. We begin by stating some notation and terminology. In this paper $\Gamma = (V, E)$ denotes a simple graph of order n and size m. The degree of a vertex $v \in V$ will be denoted by $\delta(v)$, the minimum degree will be denoted by δ, and the maximum degree by Δ. The subgraph induced by a set $S \subset V$ will be denoted by $\langle S \rangle$. For a non-empty subset $S \subset V$, and a vertex $v \in V$, we denote by $N_S(v)$ the set of neighbors that v has in S: $N_S(v) := \{u \in S : u \sim v\}$. Similarly, we denote by $N_{V\setminus S}(v)$ the set of neighbors that v has in $V \setminus S$: $N_{V\setminus S}(v) := \{u \in V \setminus S : u \sim v\}$. The boundary of a set $S \subset V$ is defined as $\partial(S) := \bigcup_{v \in S} N_{V\setminus S}(v)$.

A non-empty set of vertices $S \subset V$ is called an *offensive alliance* if and only if for every $v \in \partial(S)$, $|N_S(v)| \geq |N_{V\setminus S}(v)| + 1$. That is, a non-empty set of vertices $S \subset V$ is called an offensive alliance if and only if for every $v \in \partial(S)$, $2|N_S(v)| \geq \delta(v) + 1$.

An offensive alliance S is called *strong* if for every vertex $v \in \partial(S)$, $|N_S(v)| \geq |N_{V\setminus S}(v)| + 2$. In other words, an offensive alliance S is called strong if for every vertex $v \in \partial(S)$, $2|N_S(v)| \geq \delta(v) + 2$.

The *offensive alliance number* (respectively, *strong offensive alliance number*), denoted as $a_o(\Gamma)$ (respectively, $a_{\hat{o}}(\Gamma)$), is defined as the minimum cardinality of an offensive alliance (respectively, a strong offensive alliance) in Γ.

A non-empty set of vertices $S \subset V$ is a *global offensive alliance* if for every vertex $v \in V \setminus S$, $|N_S(v)| \geq |N_{V\setminus S}(v)| + 1$. Thus, global offensive alliances are also dominating sets, and one can define the *global offensive alliance number*, denoted as $\gamma_o(\Gamma)$, to equal the minimum cardinality of a global offensive alliance in Γ. Analogously, $S \subset V$ is a *global strong offensive alliance* if for every vertex $v \in V \setminus S$, $|N_S(v)| \geq |N_{V\setminus S}(v)| + 2$, and the *global strong offensive alliance number*, denoted as $\gamma_{\hat{o}}(\Gamma)$, is defined as the minimum cardinality of a global strong offensive alliance in Γ.

It was shown in [2] that the offensive alliance number of a graph of order $n \geq 2$ is bounded by

$$a_o(\Gamma) \leq \left\lfloor \frac{2n}{3} \right\rfloor \tag{1}$$

and

$$a_o(\Gamma) \leq \left\lfloor \frac{\gamma(\Gamma) + n}{2} \right\rfloor, \tag{2}$$

where $\gamma(\Gamma)$ denotes the domination number of Γ, and the strong offensive alliance number of a graph of order $n \geq 3$ is bounded by

$$a_{\hat{o}}(\Gamma) \leq \left\lfloor \frac{5n}{6} \right\rfloor. \tag{3}$$

It is clear that $a_o(\Gamma) \leq \gamma_o(\Gamma)$ and $a_{\hat{o}}(\Gamma) \leq \gamma_{\hat{o}}(\Gamma)$. In this paper we show a new proof technique for the above results and we obtain new bounds for $\gamma_o(\Gamma)$ and $\gamma_{\hat{o}}(\Gamma)$. We suggest that the "power" of an offensive alliance is greater if the subgraph induced by the alliance is connected than if it is not. Following this idea, the last section of this paper is devoted to the study of *connected alliances*, i.e., alliances whose induced subgraphs are connected.

2. Bounding above the global offensive alliance number

The following theorem shows upper bounds for the global alliance number. We emphasize that bounds (ii) and (iii) of Theorem 1 were already proven in [2]. Here we present a new techniques of proof for these results.

Theorem 1. *For all connected graphs Γ of order $n \geq 2$,*

(i) $\gamma_o(\Gamma) \leq \min\{n - \alpha(\Gamma), \lfloor \frac{n+\alpha(\Gamma)}{2} \rfloor\}$, *where $\alpha(\Gamma)$ denotes the independence number of Γ;*
(ii) *([2] Section 3, Theorem 1)* $\gamma_o(\Gamma) \leq \lfloor \frac{2n}{3} \rfloor$;
(iii) *([2] Section 3, Observation 11)* $\gamma_o(\Gamma) \leq \lfloor \frac{\gamma(\Gamma)+n}{2} \rfloor$, *where $\gamma(\Gamma)$ denotes the domination number of Γ;*
(iv) $\gamma_o(\Gamma) \leq \lfloor \frac{n(2\mu-\delta)}{2\mu} \rfloor$, *where μ denotes the Laplacian spectral radius*[1] *of Γ and δ denotes its minimum degree.*

[1] The Laplacian spectral radius of Γ is the largest Laplacian eigenvalue of Γ.

Proof. Let $S \subset V$ be an independent set of maximum cardinality $\alpha(\Gamma)$. Since the set $V \setminus S$ is a global offensive alliance in $\Gamma = (V, E)$, then

$$\gamma_o(\Gamma) + \alpha(\Gamma) \leq n. \tag{4}$$

If $|V \setminus S| = 1$, then $\Gamma = K_{1,n-1}$ and $\gamma_o(\Gamma) = 1$. If $|V \setminus S| \neq 1$, let $V \setminus S = X \cup Y$ be a partition of $V \setminus S$ such that the edge-cut between X and Y has the maximum cardinality. Suppose $|X| \leq |Y|$. For every $v \in Y$, $|N_S(v)| \geq 1$ and $|N_X(v)| \geq |N_Y(v)|$. Therefore, the set $W = S \cup X$ is a global offensive alliance in Γ, i.e., for every $v \in Y$, $|N_W(v)| \geq |N_Y(v)| + 1$. Then we have $2|X| + \alpha(\Gamma) \leq n$ and $\gamma_o(\Gamma) \leq |X| + \alpha(\Gamma)$. Thus,

$$2\gamma_o(\Gamma) - \alpha(\Gamma) \leq n. \tag{5}$$

The bounds (i) and (ii) follow from (4) and (5).

The proof of (iii) follows in the spirit of the proof of (5): in this case we take $S \subset V$ as a dominating set of minimum cardinality.

On the other hand, it was shown in [8] that

$$\alpha(\Gamma) \leq \frac{n(\mu - \delta)}{\mu}.$$

Thus, by (5) we obtain (iv). □

The above bounds are attained, for instance, for the cocktail-party graph[2] $\Gamma = K_{2,2,2}$ where $n = \mu = 6$, $\delta = 4$, $\alpha(\Gamma) = \gamma(\Gamma) = 2$ and $\gamma_o(\Gamma)$=4.

Notice that the bound $\frac{\gamma(\Gamma)+n}{2}$ on $\gamma_o(\Gamma)$ is never worse than the bound $\frac{n+\alpha(\Gamma)}{2}$. The advantage of the second one is when there is known the independence number but not the domination number.

In the spirit of the proof of (iii) we obtain

$$2\gamma_o(\Gamma) - \gamma_c \leq n, \tag{6}$$

where $\gamma_c(\Gamma)$ denotes the connected-domination number of Γ. Moreover, it was shown in [5] that if Γ is a connected graph of order n and maximum degree Δ, then

$$\gamma_c \leq n - \Delta. \tag{7}$$

Thus, by (6) and (7) we obtain the following result.

Observation 2. *For all connected graphs Γ of order n and maximum degree Δ,*

$$\gamma_o(\Gamma) \leq \left\lfloor \frac{2n - \Delta}{2} \right\rfloor. \tag{8}$$

This bound improves (ii) if $\Delta > \frac{2n}{3}$. Moreover, the bound (ii) can be improved for the case of bipartite graphs as follows.

Observation 3. *For all nontrivial bipartite graphs of order n,*

$$\gamma_o(\Gamma) \leq \frac{n}{2}. \tag{9}$$

The following theorem shows upper bounds for the global strong alliance number. We emphasize that bounds (iii) and (iv) of Theorem 4 are already proven in [2]. Here we present a new technique of proof for these results.

Theorem 4. *For all connected graphs Γ of order n,*

(i) $\gamma_{\hat{o}}(\Gamma) \leq \lfloor \frac{n+\gamma_2(\Gamma)}{2} \rfloor$, *where $\gamma_2(\Gamma)$ denotes the 2-domination number of Γ.*

[2] The cocktail-party graph is a graph of order 6 consisting of two rows of paired nodes in which all nodes but the paired ones are connected with an edge.

If $\delta \geq 2$, *then*

(ii) $\gamma_{\hat{o}}(\Gamma) \leq n - \alpha(\Gamma)$, *where* $\alpha(\Gamma)$ *denotes the independence number of* Γ*;*
(iii) $\gamma_{\hat{o}}(\Gamma) \leq \lfloor \frac{5n}{6} \rfloor$*;*
(iv) *If* Γ *is a 3-regular graph, then* $\gamma_{\hat{o}}(\Gamma) \leq \lfloor \frac{3n}{4} \rfloor$.

Proof. Let $H \subset V$ be a 2-dominating set of minimum cardinality. If $|V \setminus H| = 1$, then $\gamma_2(\Gamma) = n - 1$ and $\gamma_{\hat{o}}(\Gamma) \leq n - 1$. If $|V \setminus H| \neq 1$, let $V \setminus H = X \cup Y$ be a partition of $V \setminus H$ such that the edge-cut between X and Y has the maximum cardinality. Suppose $|X| \leq |Y|$. For every $v \in Y$, $|N_H(v)| \geq 2$ and $|N_X(v)| \geq |N_Y(v)|$. Therefore, the set $W = H \cup X$ is a global strong offensive alliance in Γ, i.e., for every $v \in Y$, $|N_W(v)| \geq |N_Y(v)| + 2$. Then it follows that

$$2|X| + \gamma_2(\Gamma) \leq n \tag{10}$$

and

$$\gamma_{\hat{o}}(\Gamma) \leq |X| + \gamma_2(\Gamma). \tag{11}$$

Thus, by (10) and (11), (i) follows.

Let $S \subset V$ be an independent set of maximum cardinality $\alpha(\Gamma)$. Since $\delta \geq 2$, the set $V \setminus S$ is a global strong offensive alliance in $\Gamma = (V, E)$. Hence, (ii) follows. On the other hand, it was shown in [1] that

$$\delta \geq 2 \Rightarrow \gamma_2(\Gamma) \leq \frac{2n}{3}. \tag{12}$$

So, by (i) and (12), (iii) follows.

On the other hand, if Γ is connected and 3-regular, then for all global strong offensive alliances S such that $|S| = \gamma_{\hat{o}}(\Gamma)$, $V \setminus S$ is an independent set. Thus, $\frac{3n}{2} = m \leq 3(n - \gamma_{\hat{o}}(\Gamma)) + \gamma_{\hat{o}}(\Gamma)$. Hence, (iv) follows. □

The bounds (i) and (ii) are reached, for instance, for the cocktail-party graph $\Gamma = K_6 - F$[3] where $\gamma_2(\Gamma) = 2$ and $\gamma_{\hat{o}}(\Gamma) = 4$. The bound (iii), is attained, for instance, for the left hand side graph of Fig. 1: in this case $\gamma_{\hat{o}}(\Gamma) = 6$. An example of equality in (iv) is $\Gamma = K_3 \times K_2$. We emphasize that there are graphs with minimum degree 1, such that bounds (ii) and (iii) fail. This is, for instance, the case for the star graph, $\Gamma = K_{1,r}$, with $r \geq 6$. In this case $n = r + 1$ and $\gamma_{\hat{o}}(\Gamma) = \alpha(\Gamma) = r$.

Notice that, in the case of 3-regular graphs, a set $S \subset V$ is a global strong offensive alliance if and only if S is a 3-dominating set. As a consequence, in this case, $S \subset V$ is a global strong offensive alliance if and only if $V \setminus S$ is an independent set. Therefore,

$$\gamma_{\hat{o}}(\Gamma) = n - \alpha(\Gamma). \tag{13}$$

That is, in the case of 3-regular graphs, the global strong alliance number coincides with the vertex cover number. The reader is referred to [10] for a more general study on offensive alliances in 3-regular graphs.

3. Bounding below the global offensive alliance number

Theorem 5. *For all connected graphs* Γ *of order* n*, minimum degree* δ *and maximum degree* Δ*,*

(i) $$\gamma_0(\Gamma) \geq \begin{cases} \left\lceil \dfrac{n(\delta+1)}{2\Delta+\delta+1} \right\rceil & \text{if } \delta \text{ odd;} \\ \left\lceil \dfrac{n\delta}{2\Delta+\delta} \right\rceil & \text{otherwise;} \end{cases}$$

(ii) $$\gamma_{\hat{0}}(\Gamma) \geq \begin{cases} \left\lceil \dfrac{n(\delta+3)}{2\Delta+\delta+3} \right\rceil & \text{if } \delta \text{ odd;} \\ \left\lceil \dfrac{n(\delta+2)}{2\Delta+\delta+2} \right\rceil & \text{otherwise.} \end{cases}$$

[3] F denotes a 1-factor.

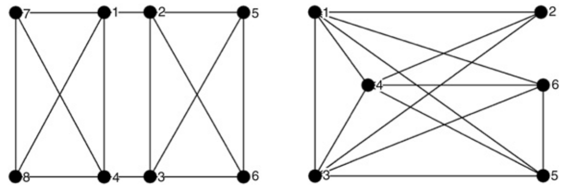

Fig. 1.

Proof. Let $\gamma_k(\Gamma)$ denote the k-domination number of Γ. Since every global offensive alliance is a $\lceil\frac{\delta+1}{2}\rceil$-dominating set and any global strong offensive alliance is a $\lceil\frac{\delta+2}{2}\rceil$-dominating set,

$$\gamma_{\lceil\frac{\delta+1}{2}\rceil}(\Gamma) \le \gamma_0(\Gamma) \tag{14}$$

and

$$\gamma_{\lceil\frac{\delta+2}{2}\rceil}(\Gamma) \le \gamma_{\hat{0}}(\Gamma). \tag{15}$$

On the other hand, for every k-dominating set $S \subset V$, $k(n-|S|) \le \Delta|S|$. Hence,

$$\gamma_k(\Gamma) \ge \left\lceil \frac{kn}{\Delta+k} \right\rceil. \tag{16}$$

Therefore, the result follows. □

Examples of equality in above theorem are $\Gamma = K_{3,3}$ and the 3-cube graph.

The following result provides tight bounds on $\gamma_o(\Gamma)$ and $\gamma_{\hat{o}}(\Gamma)$ in terms of the order and size of Γ.

Theorem 6. *For all graphs Γ of order n and size m,*

$$\gamma_o(\Gamma) \ge \left\lceil \frac{3n - \sqrt{9n^2 - 8n - 16m}}{4} \right\rceil$$

and

$$\gamma_{\hat{o}}(\Gamma) \ge \left\lceil \frac{3n + 1 - \sqrt{9n^2 - 10n - 16m + 1}}{4} \right\rceil.$$

Proof. If S denotes a global offensive alliance in $\Gamma = (V, E)$, then

$$2m = \sum_{v\in V\setminus S} \delta(v) + \sum_{v\in S} \delta(v) \le (n-|S|)(2|S|-1) + |S|(n-1). \tag{17}$$

Hence, solving $2|S|^2 - 3n|S| + 2m + n \le 0$ we obtain the bound on $\gamma_o(\Gamma)$. The bound on $\gamma_{\hat{o}}(\Gamma)$ is derived by analogy by using $2m \le (n-|S|)(2|S|-2) + |S|(n-1)$ instead of (17). □

Equality in the above bound is shown by the graph on the right in Fig. 1 where $S = \{2, 6, 5\}$ is a minimal global offensive alliance and $S' = \{1, 3, 4\}$ is a minimal global strong offensive alliance. Even so, the following bounds, expressed in terms of the order, size, and the maximum degree of Γ, improve the previous result.

Theorem 7. *For all graphs Γ of order n, size m and maximum degree Δ,*

$$\gamma_0(\Gamma) \ge \left\lceil \frac{2m+n}{3\Delta+1} \right\rceil \quad \text{and} \quad \gamma_{\hat{0}}(\Gamma) \ge \left\lceil \frac{2(m+n)}{3\Delta+2} \right\rceil.$$

Proof. If $S \subset V$, then

$$|S|\Delta \ge \sum_{v\in V\setminus S} |N_S(v)|. \tag{18}$$

Moreover, if S is a global offensive alliance in Γ, then

$$\sum_{v\in V\setminus S}|N_S(v)| \geq \sum_{v\in V\setminus S}|N_{V\setminus S}(v)| + (n-|S|). \tag{19}$$

Thus,

$$\begin{aligned} 2m &= \sum_{v\in V\setminus S}\delta(v) + \sum_{v\in S}\delta(v) \\ &= \sum_{v\in V\setminus S}|N_S(v)| + \sum_{v\in V\setminus S}|N_{V\setminus S}(v)| + \sum_{v\in S}\delta(v) \\ &\leq 2\sum_{v\in V\setminus S}|N_S(v)| + |S| - n + \sum_{v\in S}\delta(v) \\ &\leq (3\Delta+1)|S| - n. \end{aligned}$$

So, the bound on $\gamma_0(\Gamma)$ follows. If the global offensive alliance S is strong, then we have

$$\sum_{v\in V\setminus S}|N_S(v)| \geq \sum_{v\in V\setminus S}|N_{V\setminus S}(v)| + 2(n-|S|). \tag{20}$$

Basically, the bound on $\gamma_{\hat{o}}(\Gamma)$ follows as before by using (20) instead of (19). □

The above bounds are reached, for instance, in the case of the 3-cube graph $\Gamma = K_2 \times K_2 \times K_2$, where $\gamma_o(\Gamma) = \gamma_{\hat{o}}(\Gamma) = 4$. Notice that Theorem 6 only gives $\gamma_o(\Gamma) \geq 2$.

As we can see in [7], we can obtain bounds on the alliance numbers from the spectrum of Γ or from the Laplacian spectrum of Γ. For instance, the following result was proved in [7]. For completeness we include the proof of this result.

Theorem 8. *For all graphs Γ of order n, size m, minimum degree δ and Laplacian spectral radius μ,*

$$\gamma_o(\Gamma) \geq \left\lceil \frac{n}{\mu}\left\lceil\frac{\delta+1}{2}\right\rceil\right\rceil \quad \textit{and} \quad \gamma_{\hat{o}}(\Gamma) \geq \left\lceil \frac{n}{\mu}\left(\left\lceil\frac{\delta}{2}\right\rceil+1\right)\right\rceil.$$

Proof. It was shown in [3] that the Laplacian spectral radius of Γ, μ, satisfies

$$\mu = 2n\max\left\{\frac{\sum_{v_i\sim v_j}(w_i-w_j)^2}{\sum_{v_i\in V}\sum_{v_j\in V}(w_i-w_j)^2} : w \neq \alpha\mathbf{j} \text{ for } \alpha\in\mathbb{R}\right\}. \tag{21}$$

Let $S \subset V$. From (21), taking $w \in \mathbb{R}^n$ defined as

$$w_i = \begin{cases} 1 & \text{if} \quad v_i \in S; \\ 0 & \text{otherwise,} \end{cases}$$

we obtain

$$\mu \geq \frac{n\sum_{v\in V\setminus S}|N_S(v)|}{|S|(n-|S|)}. \tag{22}$$

Moreover, if S is a global offensive alliance in Γ,

$$|N_S(v)| \geq \left\lceil\frac{\delta(v)+1}{2}\right\rceil \quad \forall v \in V\setminus S. \tag{23}$$

Thus, (22) and (23) lead to

$$\mu \geq \frac{n}{|s|}\left\lceil\frac{\delta+1}{2}\right\rceil. \tag{24}$$

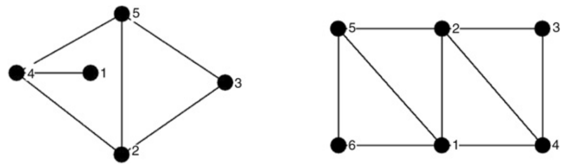

Fig. 2.

Therefore, solving (24) for $|S|$, and considering that it is an integer, we obtain the bound on $\gamma_{a_o}(\Gamma)$. If the global offensive alliance S is strong, then

$$|N_S(v)| \geq \left\lceil \frac{\delta(v)}{2} \right\rceil + 1 \quad \forall v \in V \setminus S. \tag{25}$$

Thus, (22) and (25) lead to the bound on $\gamma_{\hat{o}}(\Gamma)$. $\square$

If Γ is the Petersen graph, then $\mu = 5$. Thus, Theorem 8 leads to $\gamma_o(\Gamma) \geq 4$ and $\gamma_{\hat{o}}(\Gamma) \geq 6$. Therefore, the above bounds are tight.

4. Offensive alliances and connected subgraphs

An offensive alliance (global offensive alliance) S in Γ is *minimal* if no proper subset of S is an offensive alliance (global offensive alliance) in Γ.

Theorem 9. *Let $\Gamma = (V, E)$ be a connected graph of order n and diameter $D(\Gamma)$. If Γ has a minimal (global) offensive alliance S such that $\langle V \setminus S\rangle$ is connected, then $D(\Gamma) \leq n - |S| + 1$.*

Proof. If $S \subset V$ is a minimal (global) offensive alliance in Γ then $V \setminus S$ is a dominating set in Γ. So, if $\langle V \setminus S\rangle$ is connected, then $D(\Gamma) \leq D(\langle V \setminus S\rangle) + 2$. Hence, $D(\Gamma) \leq n - |S| + 1$. $\square$

We remark that there are graphs such that for every minimal (global) offensive alliance S, $\langle V \setminus S\rangle$ is not connected; for instance, the case of the 3-cube graph.

The above bound is tight. Let Γ be the left hand side graph of Fig. 2. In this case the set $S = \{1, 3, 5\}$ is a minimal global offensive alliance and $V \setminus S = \{2, 4\}$ is connected. Thus, $3 = D(\Gamma) \leq n - |S| + 1 = 3$.

Theorem 10. *Let $\Gamma = (V, E)$ be a graph of order n and maximum degree Δ. For all minimal global offensive alliance S such that $\langle V \setminus S\rangle$ is connected,*

$$|S| \geq \left\lceil \frac{3n - 2}{\Delta + 3} \right\rceil.$$

Moreover, for all minimal global strong offensive alliances S such that $\langle V \setminus S\rangle$ is connected,

$$|S| \geq \left\lceil \frac{4n - 2}{\Delta + 4} \right\rceil.$$

Proof. Let $S \subset V$. As $\langle V \setminus S\rangle$ is connected,

$$\sum_{v \in V \setminus S} |N_{V \setminus S}(v)| \geq 2(n - |S| - 1). \tag{26}$$

So, the first bound follows, by (18), (19) and (26). The second bound is analogously derived by using (20) instead of (19). $\square$

The above bounds are tight. If Γ is the left hand side graph of Fig. 2, then $S = \{1, 3, 5\}$ is a minimal global offensive alliance in Γ and $V \setminus S = \{2, 4\}$ is connected. Moreover, if Γ is the right hand side graph of Fig. 2, then $S = \{3, 4, 5, 6\}$ is a minimal global strong offensive alliance in Γ and $V \setminus S = \{1, 2\}$ is connected.

We define the *global-connected offensive alliance number*, $\gamma_{co}(\Gamma)$ (respectively, *global-connected strong offensive alliance number* $\gamma_{c\hat{o}}(\Gamma)$) as the minimum cardinality of any global offensive alliance (respectively, global strong offensive alliance) in Γ whose induced subgraph is connected.

Theorem 11. *Let Γ be a simple graph of order n, size m, diameter D and maximum degree Δ. The global-connected offensive alliance number of Γ is bounded by*

$$\gamma_{co}(\Gamma) \geq \left\lceil \frac{2m+n+2(D-1)^2}{2n+\Delta+1} \right\rceil$$

and the global-connected strong offensive alliance number of Γ is bounded by

$$\gamma_{\hat{c}\hat{o}}(\Gamma) \geq \left\lceil \frac{2\left(m+n+(D-1)^2\right)}{2n+\Delta+2} \right\rceil.$$

Proof. If S is a global offensive alliance in $\Gamma = (V, E)$, then by (19) we have

$$(|S|-1)(n-|S|) \geq \sum_{v \in V \setminus S} |N_{V \setminus S}(v)|. \tag{27}$$

Thus,

$$(2|S|-1)(n-|S|) \geq \sum_{v \in V \setminus S} |N_S(v)| + \sum_{v \in V \setminus S} |N_{V \setminus S}(v)| = \sum_{v \in V \setminus S} \delta(v). \tag{28}$$

Therefore,

$$(2|S|-1)(n-|S|) + \Delta|S| \geq \sum_{v \in V \setminus S} \delta(v) + \sum_{v \in S} \delta(v) = 2m. \tag{29}$$

On the other hand, if S is a dominating set and $\langle S \rangle$ is connected, then $D(\Gamma) \leq D(\langle S \rangle) + 2$. So, $D(\Gamma) \leq |S| + 1$. Hence,

$$2n|S| - n + |S| + \Delta|S| \geq 2m + 2(D(\Gamma)-1)^2. \tag{30}$$

Thus, the bound on $\gamma_{co}(\Gamma)$ follows. Basically the bound on $\gamma_{\hat{c}\hat{o}}(\Gamma)$ follows as before by using (20) instead of (19). □

The above bounds are tight, as we show in the following instance. Let $\Gamma_{3,t}$ be the graph obtained by joining every vertex of the complete graph K_3 to every vertex of the trivial graph of order $t \geq 8$. In such a case, $\gamma_{co}(\Gamma_{3,t}) = \gamma_{\hat{c}\hat{o}}(\Gamma_{3,t}) = 3$ and Theorem 11 leads to $\gamma_{co}(\Gamma_{r,t}) \geq 3$ and $\gamma_{\hat{c}\hat{o}}(\Gamma_{3,t}) \geq 3$.

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